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Note

Non-density of Restricted Rationals

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In [2] the author introduced restricted rational functions of the form $p(x)/q(x)$, where

$$0 < \mu(x) \leqq q(x) \leqq \nu(x) \tag{1}$$

and μ and ν are continuous.

A special case was given earlier by Krabs, and a problem implying restraints of that type, by Kaufman and Taylor. We consider the density of such rationals in $C[\alpha, \beta]$ for μ, ν *fixed* but otherwise arbitrary. Providing there is a polynomial q satisfying the restraints, density follows from the Weierstrass theorem if p is allowed to be freely chosen. We consider the case in which n is given, p runs over all polynomials of degree $\leqq n$ and we approximate a function f with no zeros. It is well-known that f can be approximated arbitrarily well by $1/q$, q unrestricted but >0 or <0. See [1] for such results. Assume $f > 0$.

First consider the case in which p is taken from the constants. Without loss of generality we consider the case in which $\mu = \varepsilon$ (constant) and $\nu = 1$. Then any restricted $r > 0$ satisfies

$$\frac{\min r(x)}{\max r(x)} \leqq \frac{1}{\varepsilon}$$

and if $\min f(x)/\max f(x) > 1/\varepsilon$, f cannot be approximated.

Next let p be of fixed degree n and assume p/q can approximate any $f > 0$ arbitrarily well. We first set $f = 1$ on the left side S of the interval. Then any p/q closely approximating f nearly satisfies the inequality

$$\mu(x) \leqq p(x) \leqq \nu(x), \qquad x \in S$$

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so that for some small δ,

$$\mu(x) - \delta \leq p(x) \leq \nu(x) + \delta, \qquad x \in S. \tag{2}$$

But (2) implies by [4, p. 24] that the l_p norm of the coefficients of p is bounded, say by M. Hence

$$p(x) \leq M \sum_{k=0}^{n} |x^k| = N, \qquad \alpha \leq x \leq \beta.$$

Hence $p(x) \leq N/\mu(x)$, and if we select f larger at β we do not get approximation there.

The L_p norm on $[\alpha, \beta]$ is also of interest. The argument for constant numerator carries over. By an argument of Tornheim [5, Theorem 1], p/q approximating 1 on S implies that the l_p norm of the coefficients of p is bounded and the argument for general p goes through.

Should we require, as in some of [3], that the set Q_c of allowable denominators q exclude denominators with zeros and be compact, then there must exist μ, ν such that (1) holds and consequently our non-density result holds.

Results indicating which functions can be as closely approximated by restricted rationals as by non-restricted rationals would be welcome.

A generalization of rationals consists of "powered rationals" p^s/q^r, s and r being fixed natural numbers. It is seen that if p is of fixed degree and q is restricted as above, restricted powered rationals are non-dense by similar arguments.

References

1. B. Boehm, Convergence of best rational Tchebycheff approximations, *Trans. Amer. Math. Soc.* **115** (1965), 388–399.
2. C. Dunham, Chebyshev approximation by rationals with constrained denominators, *J. Approx. Theory* **37** (1983), 5–11.
3. C. Dunham, Chebyshev approximation by restricted rationals, *J. Approx. Theory Appl.* **1** (1985), 111–118.
4. J. Rice, "The Approximation of Functions," Vol. 1, Addison-Wesley, Reading, MA, 1964.
5. L. Tornheim, Approximation by families of functions, *Proc. Amer. Math. Soc.* **7** (1956), 641–643.

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